Recent progress on anomaly matching (especially for QCD and spin systems)

Yuya Tanizaki

RIKEN BNL Research Center, Brookhaven National Laboratory

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Outline

- Main message
- General theorem (anomaly matching)
- Examples
  - Projective rep. in quantum mechanics
  - Chiral anomalies in even dimensions
  - 4-dim. $SU(N)$ Yang-Mills theory at $\theta = \pi$
- Applications (QCD, spin systems)

Message

Quantum field theorists are interested in low-energy behaviors:

Problem

Does a QFT have the unique gapped ground state?

Both answers (Yes and No) to this question are equally interesting.
On the other hand, I want to convince that

- It is almost always HARD to answer YES.
- It is often EASY to answer NO.

General theorem (Anomaly matching)

Symmetry of QFT

Let me clarify what I call symmetry in this talk:

Definition

QFT has (0-form) global symmetry G if

- *G maps physical local operators to physical local operators.*
- *For any $g \in G$ with $g \neq 1$, some physical local operator $\mathcal{O}(x)$ satisfies $g \cdot \mathcal{O}(x) \neq \mathcal{O}(x)$ (i.e. G has a faithful rep.).*

Comments:

- Gauge redundancy is not symmetry, as it violates the 2nd condition.

[Generalization to higher-symmetries (cf. Gaiotto, Kapustin, Seiberg, Willet, '14)]

't Hooft anomaly

Let us define the 't Hooft anomaly:

Definition

Symmetry G of QFT has an 't Hooft anomaly if the partition function $\mathcal{Z}_d[A]$ with the background G-gauge field A is anomalous:

$$\mathcal{Z}_d[A + \mathrm{d}_A\theta] = \mathcal{Z}_d[A] \exp\left(\mathrm{i}\mathcal{A}_d[\theta, A]\right),$$

with $\mathcal{A}_d[\theta, A] \neq \delta_\theta$(any d-dim. local functional of A).

Comments:

- When we want to dynamically gauge G, the anomaly is called gauge anomaly.

Anomaly matching

Theorem

We consider the RG flow: QFT$_{UV}$ $\to$ QFT$_{IR}$. Then,

$$\mathcal{A}_{d,\mathrm{UV}}[\theta, A] = \mathcal{A}_{d,\mathrm{IR}}[\theta, A].$$

This is called the 't Hooft anomaly matching condition ('t Hooft, '80, ...).

Let us show this statement under the anomaly-inflow assumption (Callan, Harvey, '85), which is empirically always true (in my knowledge):

Assumption

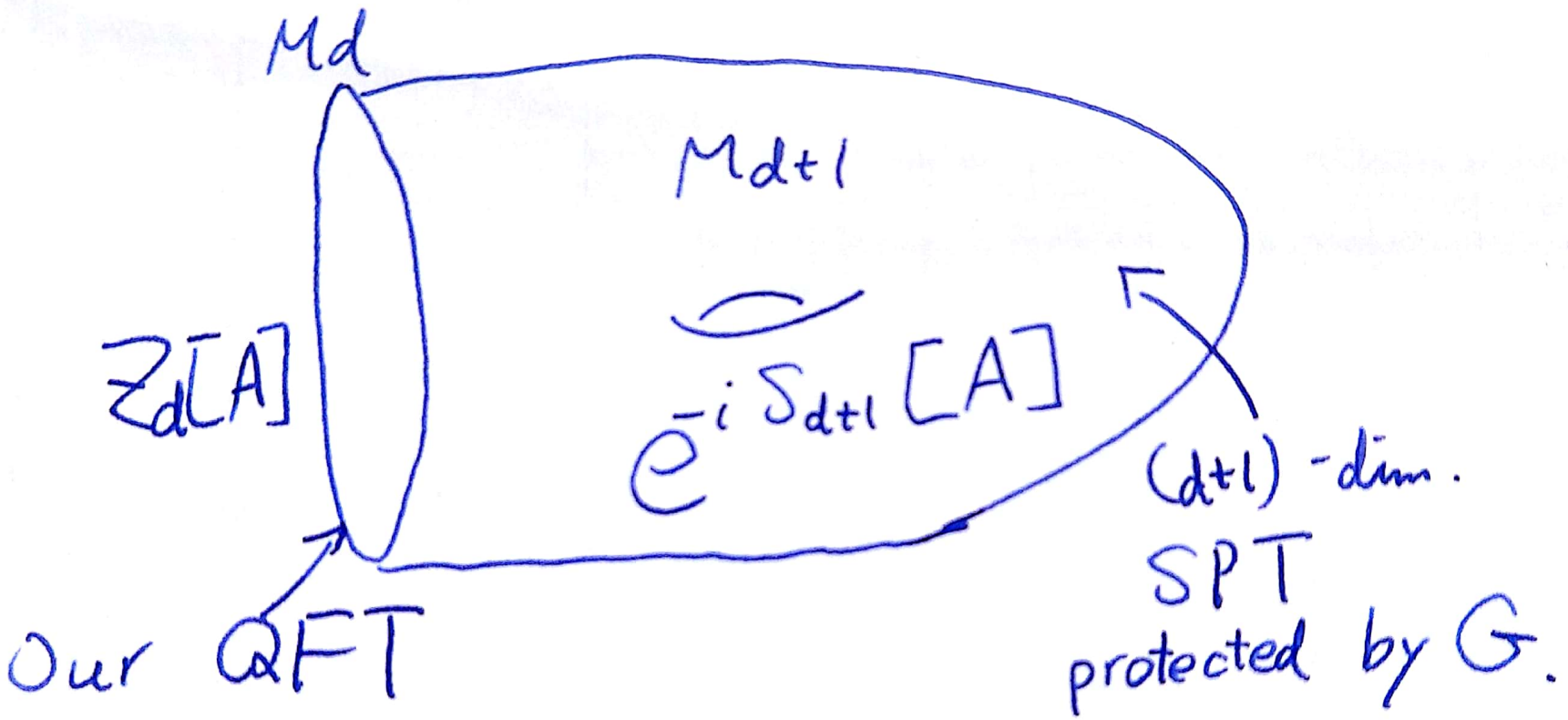
Anomaly $\mathcal{A}_d[\theta, A]$ in d-dim. satisfies

$$\mathcal{A}_d[\theta, A] = \delta_\theta(S_{d+1}[A]),$$

where $S_{d+1}[A]$ defines a $(d+1)$-dim. topological G-gauge theory.

Anomaly inflow

When gauging G, we regards our theory as a boundary of $(d+1)$-dim. symmetry-protected topological (SPT) phase protected by G:



(Wen, '13, Kapustin, Thorngren, '14, Cho, Teo, Ryu, '14, ...)

Proof of anomaly matching

RG flow, $\mathrm{QFT}_{UV} \to \mathrm{QFT}_{IR}$, means that, as we let M_d larger,

$$\mathcal{Z}_{d,\mathrm{UV}} \to \mathcal{Z}_{d,\mathrm{IR}},$$

since the contribution from massive particle excitations drop out.

Anomaly-inflow assumption states that

$$\mathcal{Z}_{d,\mathrm{UV}}[A] \exp(-\mathrm{i}S_{d+1}[A])$$

is gauge invariant with $\partial M_{d+1} = M_d$. As $M_d \to \infty$, we get

$$\mathcal{Z}_{d,\mathrm{UV}}[A] \exp(-\mathrm{i}S_{d+1}[A]) \to \mathcal{Z}_{d,\mathrm{IR}}[A] \exp(-\mathrm{i}S_{d+1}[A]).$$

Proof of anomaly matching

The gauge invariance of the combined system with the limit $M_d \to \infty$ shows

$$\begin{aligned} 0 &= \delta_\theta(\ln\{\mathcal{Z}_{d,\mathrm{UV}}[A]\exp(-\mathrm{i}S_{d+1}[A])\}) \\ &\to \delta_\theta(\ln\{\mathcal{Z}_{d,\mathrm{IR}}[A]\exp(-\mathrm{i}S_{d+1}[A])\}) \\ &= \mathrm{i}(\mathcal{A}_{d,\mathrm{IR}} - \mathcal{A}_{d,\mathrm{UV}}). \end{aligned}$$

We therefore obtain the anomaly matching condition

$$\mathcal{A}_{d,\mathrm{UV}} = \mathcal{A}_{d,\mathrm{IR}}.$$

Results of anomaly matching

Anomaly matching claims that QFT_{IR} must also have a nontrivial 't Hooft anomaly of symmetry G.

This excludes unique gapped ground states.

Possible scenarios for IR physics are

- Gappless,
- Ground states degeneracy by spontaneous breaking,
- Ground states degeneracy by topological order,

or their combinations.

Examples 1. Quantum mechanics on a circle

Quantum mechanics on S^1

We consider quantum mechanics of particle $x \in S^1$, i.e. $x \sim x + 2\pi$.
Lagrangian:

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{\theta}{2\pi}\dot{x}.$$

Canonical momentum is given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x} + \frac{\theta}{2\pi}.$$

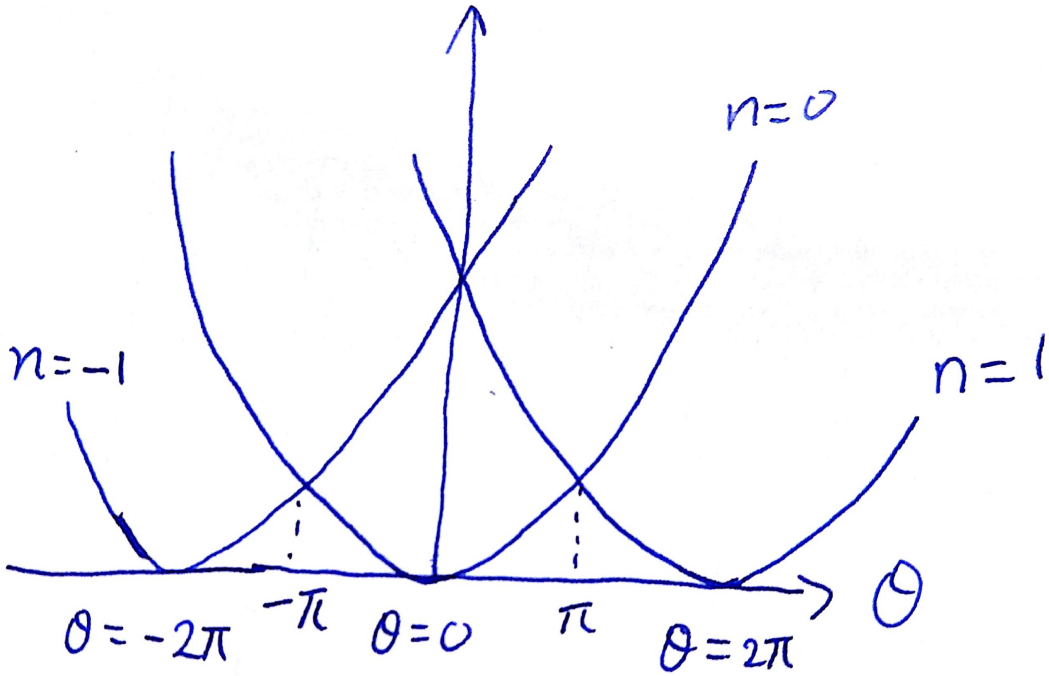
We obtain the Hamiltonian

$$H = \frac{1}{2}\left(p - \frac{\theta}{2\pi}\right)^2.$$

Quantum mechanically, $E_n = \frac{1}{2}(n - \frac{\theta}{2\pi})^2$ with the eigenstate e^{inx}.

Degeneracy of states at $\theta = \pi$

Energy spectrum looks as



- At generic θ, the ground states are unique. (At $\theta = 0$, the Feynman kernels are positive and Perron-Frobenius ensures it.)
- At $\theta = \pi$, the ground states are doubly degenerate.
  ($\Leftarrow$ Anomaly matching (Gaiotto, Kapustin, Komargodski, Seiberg, '17))

Anomaly of $U(1) \rtimes \mathbb{Z}_2$ at $\theta = \pi$

At $\theta = \pi$, the system has

- $U(1)$ symmetry, $x \mapsto x + \alpha$, and
- $\mathbb{Z}_2$ charge conjugation, $x \mapsto -x$.

Let us introduce the $U(1)$ gauge field A by minimal coupling, then

$$S[A] = \int \mathrm{d}\tau \frac{1}{2}(\dot{x} + A_0)^2 - \frac{\mathrm{i}\theta}{2\pi} \int (\mathrm{d}x + A).$$

Charge conjugation C at $\theta = \pi$ gives

$$C : S[A] \to S[A] - \mathrm{i} \int (\mathrm{d}x + A) = S[A] - \mathrm{i} \int A \ \ (\mathrm{mod} 2\pi).$$

This gives the 't Hooft anomaly:

$$C : \mathcal{Z}_{\text{1d QM},\theta=\pi}[A] \mapsto \mathcal{Z}_{\text{1d QM},\theta=\pi}[A] \mathrm{e}^{-\mathrm{i}\int A}.$$

Anomaly inflow

In order to get C invariance with A, we need to add $+\frac{\mathrm{i}}{2}\int A$ to the Lagrangian, but it breaks $U(1)$ gauge invariance.

In order to establish both invariance, we must consider 2-dim. bulk term,

$$\frac{1}{2}\int_{M_2}\mathrm{d}A,$$

and then the combined system

$$\mathcal{Z}_{1\mathrm{d\ QM}}[A]\exp\left(\frac{\mathrm{i}}{2}\int_{M_2}\mathrm{d}A\right)$$

is both $U(1)$ gauge invariant and C-invariant.

- Anomaly can show that the degeneracy at $\theta=\pi$ is persistent under any local perturbations preserving $\mathbb{Z}_2\rtimes\mathbb{Z}_2\subset U(1)\rtimes\mathbb{Z}_2$.

Examples 2. General projective representations of quantum mechanics

Projective representation of Quantum Mechanics

In quantum mechanics, states are characterized by $\mathcal{H}/\mathbb{C}^\times$ (not $\mathcal{H}$), i.e. overall phase cannot be observed.

Let G be symmetry, then $G \times \mathcal{H} \to \mathcal{H}$, $(g, \Psi) \mapsto \rho(g)\Psi$:

$$\rho(gh) = \mathrm{e}^{\mathrm{i}\phi(g,h)}\rho(g)\rho(h),$$

and we call it projective representation of G. Uniqueness of Ψ says

$$\phi(g, hk) + \phi(h, k) = \phi(gh, k) + \phi(g, h) \bmod 2\pi.$$

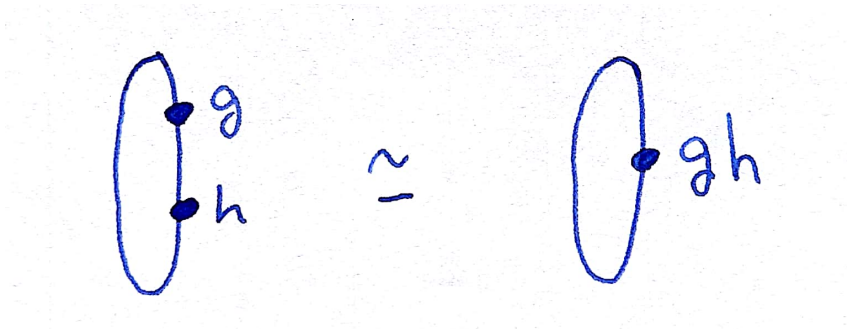
$\rho(g) \mapsto \rho(g)\mathrm{e}^{\mathrm{i}\varphi(g)}$ gives $\phi(g, h) \sim \phi(g, h) + \varphi(g) + \varphi(h) - \varphi(gh)$.

$$[\phi] \in H^2(BG, U(1)).$$

BG: classifying space ($\pi_1(BG) = G, \pi_{\geq 2}(BG) = 0$ for $|G| < \infty$).

Anomaly in 1-dim. = Projective representation

As a G-bundle on S^1, the following are gauge equivalent:



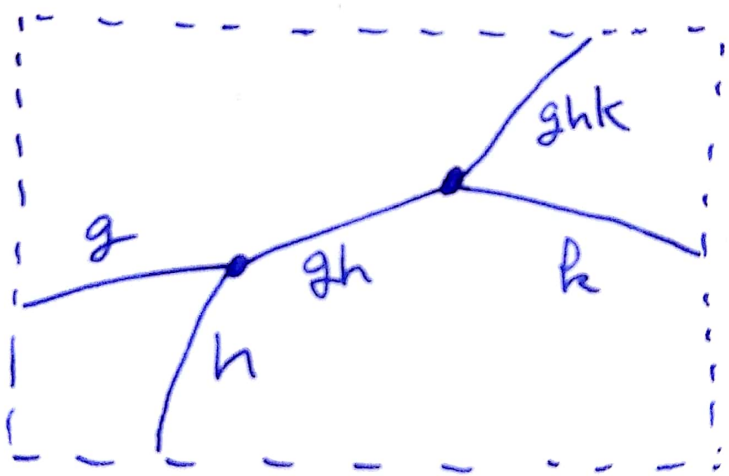
However, the partition functions are different for projective representations:

$$\mathrm{tr}[\rho(g)\rho(h)] = \mathrm{e}^{-\mathrm{i}\phi(g,h)}\mathrm{tr}[\rho(gh)] \neq \mathrm{tr}[\rho(gh)].$$

Projective phase $[\phi] \in H^2(BG, U(1))$ determines the anomaly.

2-dim. Dijkgraaf-Witten theory

Principal G-bundle (i.e. G-gauge field configuration) is characterized by the network of G-charge walls:



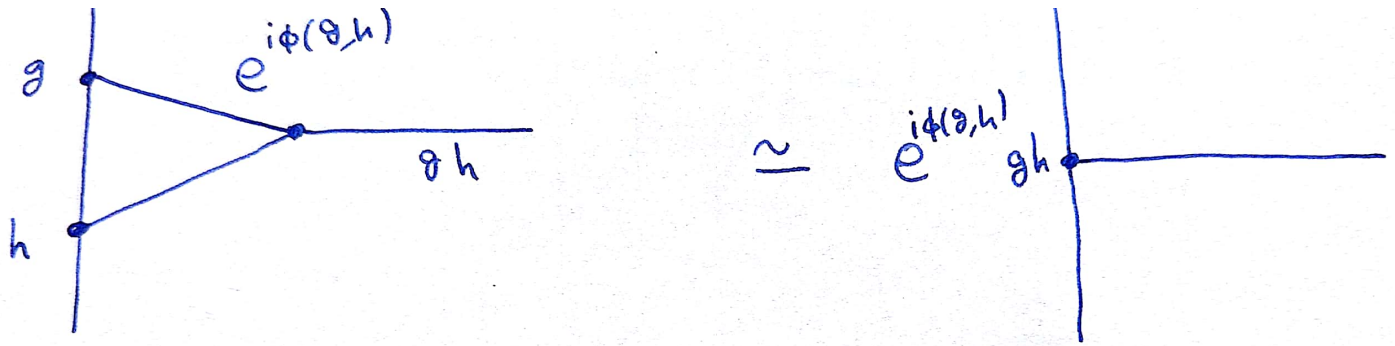
For each junction, we assign $\mathrm{e}^{\mathrm{i}\phi(g,h)}$ with $[\phi] \in H^2(BG, U(1))$, and define the action as (Dijkgraaf, Witten, '90)

$$S[A] = \sum_{\text{junctions}} \phi(g,h).$$

This defines 2d topological G-gauge theory.

Inflow from 2d DW to QM with Projective rep.

Regarding the QM with projective rep. $[\phi]$ as a boundary theory of 2d DW theory, we can establish the G-gauge invariance:



That is,

$$\mathcal{Z}_{\text{1d QM}}[A]\exp(-\mathrm{i}S_{\text{2d DW}}[A])$$

is gauge invariant.
Anomaly matching argument shows degeneracy of ground states, which reproduces the following result:

- Projective representation does not have 1-dim. representation.

Examples 3. (Perturbative) Chiral anomaly in even dimensions

(Perturbative) chiral anomaly

Let's consider left-handed fermions in $2n$ dimension:

$$\overline{\psi}\gamma_\mu\partial_\mu P_{\mathrm{L}}\psi.$$

If it has $SU(N)$ symmetry, then we consider the minimal coupling to the $SU(N)$ gauge field A,

$$\overline{\psi}\gamma_\mu(\partial_\mu + A_\mu)P_{\mathrm{L}}\psi,$$

and define the partition function $\mathcal{Z}_{2n}[A]$. Chiral anomaly is defined by its gauge transformation as

$$\mathcal{A}_{2n}[\theta, A] = -\mathrm{i}\delta_\theta \ln \mathcal{Z}_{2n}[A].$$

- Regarding θ as a FP ghost, δ_θ is BRST transformation.
- This is called consistent anomaly.

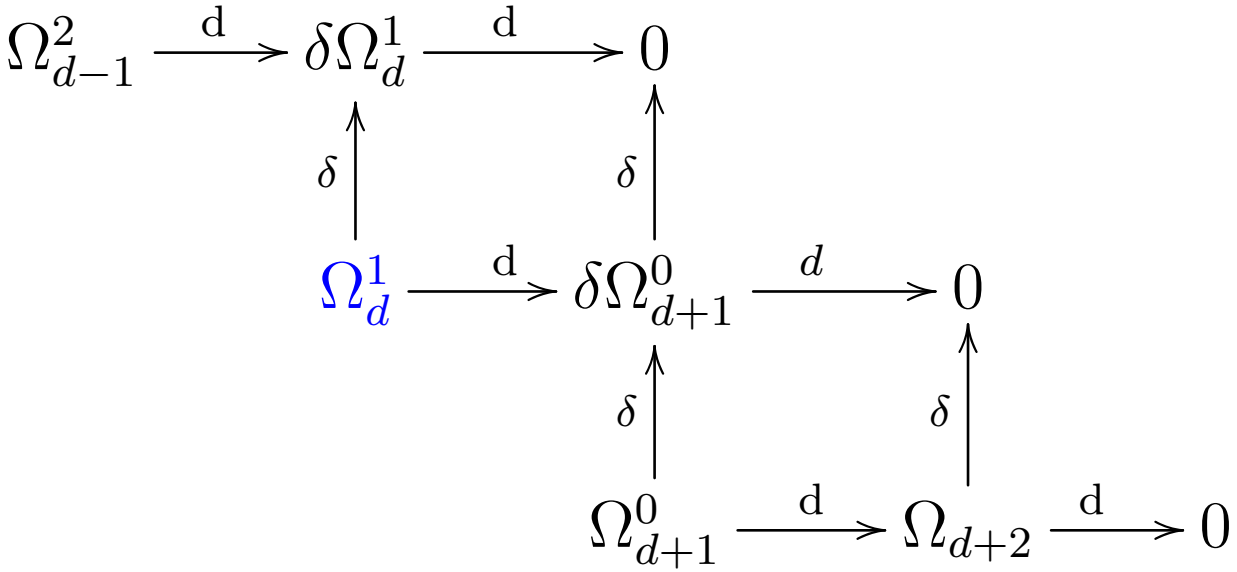
Wess-Zumino condition and Stora-Zumino chain

Since BRST derivative is nilpotent, $\delta_\theta^2 = 0$, we get

$$\delta_\theta \mathcal{A}_{2n}[\theta, A] = -\mathrm{i}\delta_\theta^2 \ln \mathcal{Z}_{2n}[A] = 0.$$

This is the Wess-Zumino consistency condition (Wess, Zumino, '71).

This can be solved in *ad hoc* way by Stora-Zumino chain (Stora, '83, Zumino, '83, Alvarez-Gaume, Ginsparg, '85): $\mathcal{A}_{2n} = \int \Omega^1_{2n}(\theta, A)$.



Anomaly inflow from $(2n+1)$-dim. Chern-Simons

With explicit computations, one can confirm that Non-Abelian anomaly satisfies the Stora-Zumino chain with

$$\Omega_{2n+2}(A) = \frac{1}{(n+1)!(2\pi)^n}\mathrm{tr}[(\mathrm{d}A+A^2)^{n+1}],\ \Omega^0_{2n+1}(A) = \mathrm{CS}_{2n+1}(A).$$

This shows that non-Abelian chiral anomaly satisfies anomaly-inflow assumption:

$$\mathcal{Z}_{2n}[A]\exp\left(-\mathrm{i}\int_{M_{2n+1}}\mathrm{CS}_{2n+1}[A]\right)$$

is gauge invariant.

Anomaly matching in QCD

QCD with $N_{\rm f}$ massless quarks has $SU(N_{\rm f})_{\rm L} \times SU(N_{\rm f})_{\rm R}$ chiral symmetry with 't Hooft anomaly:

$$\mathcal{Z}_{\text{4d QCD}}[A_{\rm L}, A_{\rm R}] \exp\left(\mathrm{i}N_{\rm c}\left\{\mathrm{CS}_5[A_{\rm R}] - \mathrm{CS}_5[A_{\rm L}]\right\}\right).$$

If the low-energy excitations are given by color-singlet particles, they must contain ('t Hooft, '80, Frishman, Schwimmer, Banks, Yankielowicz, '81, ...)

- massless fermions, or
- Nambu-Goldstone bosons.

Remarks:

- For any $N_{\rm c}$ with $N_{\rm f} > 2$, the NG bosons can always match the anomaly by Wess-Zumino term. (Wess, Zumino, '71)
- For $\mathcal{N} = 1$ SQCD with $N_{\rm f} = N_{\rm c} + 1$, the s-confining phase realize the first case. (Seiberg, '94)

Examples 4. Dashen phenomenon of $SU(N)$ Yang-Mills at $\theta = \pi$

4d $SU(N)$ pure Yang-Mills theory

We consider the $SU(N)$ Yang-Mills theory with the Lagrangian

$$S = \frac{1}{g^2}\int \mathrm{tr}[G \wedge \star G] + \frac{\mathrm{i}\theta}{8\pi^2}\int \mathrm{tr}[G \wedge G],$$

where $G = \mathrm{d}a + a^2$ is the field strength. $(\theta \sim \theta + 2\pi)$

4d $SU(N)$ YM theory has the following symmetries;

- $\mathbb{Z}_N$ one-form symmetry (center symmetry), and
- CP symmetry if $\theta = 0$ or π.

It has been widely believed that CP at $\theta = \pi$ is spontaneously broken (Dashen, '71, Coleman, '76, Witten, '80, ...).

One-form symmetry

Ordinary (i.e. 0-form) symmetry G transforms the 0-dimensional operator,

$$\mathcal{O}(x) \mapsto \rho(g)\mathcal{O}(x).$$

1-form symmetry transforms the line operators (like Wilson lines), instead of point-like operators,

$$W(C) \mapsto \mathrm{e}^{2\pi \mathrm{i}/N} W(C).$$

Generally, q-form symmetry is given as (Gaiotto, Kapustin, Seiberg, Willet, '14)

- $(d-q-1)$-dim. topological operator $U_g(M_{d-q-1})$ following a group law of G,
- q-dimensional charged operator $\mathcal{O}(C_q)$,
- they satisfy $U_g(M_{d-q-1})\mathcal{O}(C_q) = g(\mathcal{O})^{\mathrm{link}(M_{d-q-1},C_q)}\mathcal{O}(C_q)$.

(Higher-form symmetry is always Abelian)

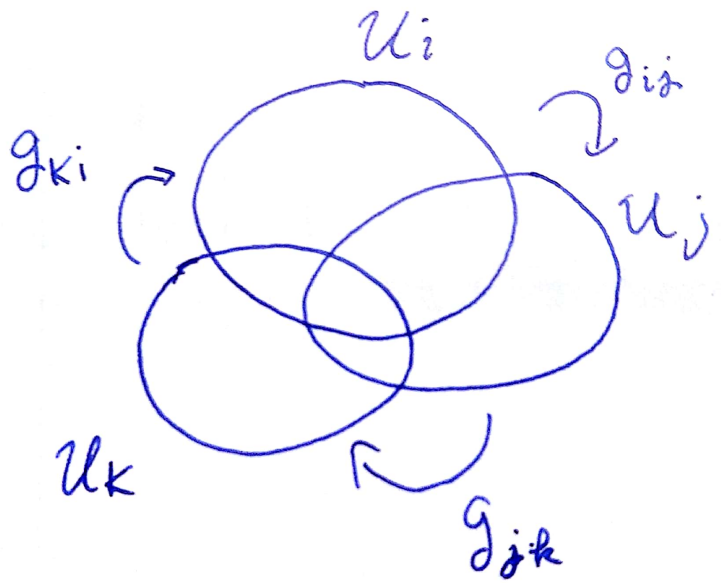
$\mathbb{Z}_N$ 1-form symmetry in 4d $SU(N)$ YM

In the case of $SU(N)$ YM, center symmetry can be realized as transformation on transition functions: $g_{ij} \mapsto g_{ij} \exp\left(\frac{2\pi \mathrm{i}}{N} k_{ij}\right)$.

Gauge field a is the collection of one-forms a_i on local patch $\mathcal{U}_i$, with the connection formula

$$a_j = g_{ij}^{-1} a_i g_{ij} + g_{ij}^{-1} \mathrm{d} g_{ij}.$$

Transition functions satisfy $g_{ki} g_{jk} g_{ij} = 1 \in SU(N)$ on triple overlap:



Gauging $\mathbb{Z}_N$ 1-form symmetry

We want to promote the 1-form symmetry $g_{ij} \mapsto g_{ij} \exp(\frac{2\pi \mathrm{i}}{N} k_{ij})$ to the gauge redundancy.

To that end, we introduce $n_{ijk} \in \mathbb{Z}_N$ on triple overlaps $\mathcal{U}_{ijk}$, and replace the cocycle condition as

$$g_{ki} g_{jk} g_{ij} = \exp\left(\frac{2\pi \mathrm{i}}{N} n_{ijk}\right).$$

Then, the above transformation on the transition function can be compensated by $n_{ijk} \mapsto n_{ijk} + k_{ij} + k_{jk} + k_{ki}$, i.e. the center symmetry is now gauge redundancy.

$$B^{(2)} = [\{n_{ijk}\}] \in H^2(M_4, \mathbb{Z}_N).$$

Continuum description of the gauged action

There is a continuum description of the center-gauged YM action (Kapustin, Seiberg, '14).

Consider YM with the gauge group $U(N) = [U(1) \times SU(N)]/\mathbb{Z}_N$: $U(N)$ gauge field $\tilde{a}$.

To eliminate the $U(1)$ photon, introduce the $U(1)$ two-form gauge field $B^{(2)}$, and set the constraint

$$N B^{(2)} = \mathrm{d}(\mathrm{tr}[\tilde{a}]).$$

This is invariant under $U(1)$ one-form gauge transformation,

$$B^{(2)} \mapsto B^{(2)} + \mathrm{d}\lambda^{(1)}, \ \tilde{a} \mapsto \tilde{a} + \lambda^{(1)} \mathbf{1}_N,$$

and we postulate this invariance in minimal coupling.
$\Rightarrow$ YM with gauge group $SU(N)/\mathbb{Z}_N$.

Anomaly of YM at $\theta = \pi$

We now obtain the partition function with the background center gauge field B:

$$\mathcal{Z}_{\rm YM}[B] = \int \mathcal{D}a \exp\left(-\frac{1}{g^2}\int |\tilde{G} - B|^2 + \frac{\mathrm{i}\theta}{8\pi^2}\int \mathrm{tr}[(\tilde{G}-B)^2] \right).$$

Performing the CP transformation, we get at $\theta = 0, \pi$, respectively,

$$\mathrm{CP}_{(\theta=0)}: \quad \mathcal{Z}_{\rm YM}[B] \mapsto \mathcal{Z}_{\rm YM}[B],$$

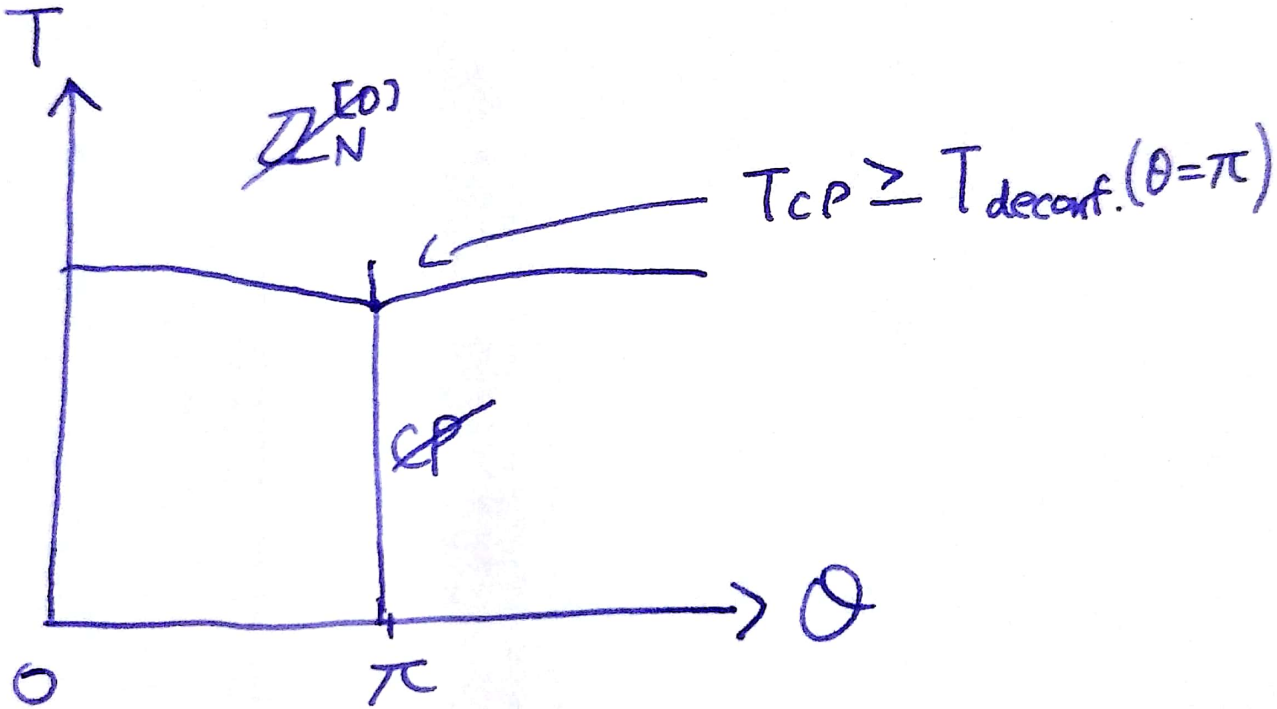
$$\mathrm{CP}_{(\theta=\pi)}: \quad \mathcal{Z}_{\rm YM}[B] \mapsto \mathcal{Z}_{\rm YM}[B] \exp\left(\frac{\mathrm{i}N}{4\pi}\int B\wedge B \right).$$

This concludes that

- For even N, $\mathbb{Z}_N \rtimes (\mathbb{Z}_2)_{\rm CP}$ is 't Hooft anomalous. (Gaiotto, Kapustin, Komargodski, Seiberg, '17)
- For odd N, no anomaly due to a counter term $\frac{N(N+1)/2}{4\pi}\int B^2$, but this gives the global inconsistency condition between $\theta = 0, \pi$. (GKKS, '17, Tanizaki, Kikuchi, '17)

Phase diagram of $SU(N)$ YM

Anomaly constrains the possible phase diagram (Gaiotto, Kapustin, Komargodski, Seiberg, '17):



The CP domain wall at $\theta = \pi$ supports $SU(N)_1$ Chern-Simons theory by anomaly-inflow. (GKKS, '17, cf. Anber, Poppitz, Sulejmanpasic, '15)

Applications 1. Discrete anomaly of QCD and Stern phase

(YT, 1807.07666)

Chiral symmetry breaking of QCD

In the massless quark limit, QCD has chiral symmetry,

$$G = \frac{SU(N_{\rm f})_{\rm L} \times SU(N_{\rm f})_{\rm R} \times U(1)_{\rm V}}{\mathbb{Z}_{N_{\rm c}} \times \mathbb{Z}_{N_{\rm f}}}.$$

If $N_{\rm f}$ is not so large, quark bilinear condensate appears,

$$\langle \overline{q_{\rm R}} q_{\rm L} \rangle = -\Lambda^3,$$

and it breaks this symmetry spontaneously,

$$G \to H = \frac{SU(N_{\rm f})_{\rm V} \times U(1)_{\rm V}}{\mathbb{Z}_{N_{\rm c}} \times \mathbb{Z}_{N_{\rm f}}}.$$

Exotic chiral symmetry breaking

Theoretically, it is also possible to consider very similar but different pattern of chiral symmetry breaking (Stern, 97, 98):

- Order parameter is quark quartic:

$$\sum_a (\overline{q_{\rm R}} T^a q_{\rm L})(\overline{q_{\rm L}} T^a q_{\rm R}).$$

  The symmetry breaking pattern is

$$G \to \frac{SU(N_{\rm f})_{\rm V} \times U(1)_{\rm V}}{\mathbb{Z}_{N_{\rm c}} \times \mathbb{Z}_{N_{\rm f}}} \times (\mathbb{Z}_{N_{\rm f}})_L.$$

- This is ruled out at the zero-density QCD, i.e., when QCD inequality is valid. (Kogan, Kovner, Shifman, 98)

Purpose

We rule out the exotic scenario based only on symmetry and anomaly.

Symmetry of massless QCD

QCD Lagrangian:

$$S = \frac{1}{2g^2}\int \mathrm{tr}(G \wedge *G) + \int \sum_{f=1}^{N_F} \overline{q}_f \gamma_\mu D_\mu q.$$

Symmetry of massless QCD (i.e. Lagrangian + $\mathcal{D}\overline{q}\mathcal{D}q$):

$$G = \frac{SU(N_F)_L \times SU(N_F)_R \times U(1)_V}{\mathbb{Z}_{N_C} \times (\mathbb{Z}_{N_F})_V}$$

- Due to the gauge invariance, symmetry must be divided by $\mathbb{Z}_{N_C} \subset SU(N_C) \times U(1)_V$.
- In other words, correct $U(1)$ symmetry is $U(1)_B = U(1)_V/\mathbb{Z}_{N_c}$ but not $U(1)_V$.

Further comments on chiral symmetry

$U(1)_A$ is explicitly broken to $(\mathbb{Z}_{2N_F})_A$ by instantons (or quantum anomaly).

This is a subgroup of the continuous flavor symmetry:

$$(\mathbb{Z}_{2N_F})_A \subset SU(N_F)_L \times SU(N_F)_R \times U(1)_V.$$

Indeed, the generator of $(\mathbb{Z}_{2N_F})_A$ can be written as

$$\mathrm{e}^{\frac{2\pi}{2N_F}\mathrm{i}\gamma_5}\mathbf{1}_{N_F} = \exp\left(\frac{2\pi}{N_F}\mathrm{i}\frac{1+\gamma_5}{2}\mathrm{diag}[1,\ldots,1,1-N_F]\right)\mathrm{e}^{-2\pi\mathrm{i}/(2N)}$$

Quick message: SSB of $(\mathbb{Z}_{2N_F})_A$ implies continuous ChSB, but the converse is not true.

Contents of background gauge fields

To emphasize the role of discrete axial symmetry, we look at

$$G^{\rm sub} = \frac{SU(N_F)_V \times U(1)_V}{(\mathbb{Z}_{N_{\rm c}}) \times (\mathbb{Z}_{N_{\rm f}})} \times (\mathbb{Z}_{N_{\rm f}})_L.$$

To detect the 't Hooft anomaly of $G^{\rm sub}$, we introduce the background gauge fields (1710.08923, 1711.10487) (ref. Kapustin, Seiberg (2014));

- $SU(N_F)_V$ one-form gauge field: A_f
- $U(1)_V$ one-form gauge field: A_V
- $(\mathbb{Z}_{N_{\rm f}})_L$ one-form symmetry: A_χ
- $(\mathbb{Z}_{N_C})$ two-form gauge field: B_c
- $(\mathbb{Z}_{N_F})$ two-form gauge field: B_f

Role of two-form gauge fields

Why do we need two-form gauge fields?

Gauging $U(1)_V/\mathbb{Z}_{N_c}$

Gauge group $SU(N_c)$ and the $U(1)_V$ phase rotation of quark have the common elements $\mathbb{Z}_{N_c}$ since q has the charge $(\square, 1)$.

$$\begin{aligned} & \text{Gauging } U(1)_B = U(1)_V/\mathbb{Z}_{N_c} \\ \Leftrightarrow \quad & U(1)_V \text{ gauge field} + \text{constraint: possible charges are } (\square^k, k). \end{aligned}$$

B_c gives the constraint on the possible gauge charges correctly!
The $U(1)_B$ gauge field A_B is given by

$$\mathrm{d}A_B = N_c(\mathrm{d}A_V + B_c).$$

Discrete 't Hooft anomaly for massless QCD

After introducing the background gauge fields, the partition function $\mathcal{Z}_{\rm QCD}$ is no longer gauge invariant.

Gauge invariance requires to add 5d SPT phase:

$$\mathcal{Z}_{\rm QCD} \exp\left(\frac{N_{\rm f}}{(2\pi)^2} \int A_\chi \wedge {\rm d}A_B \wedge B_{\rm f} \right).$$

This says that the baryon number conservation is anomalously violated under $(\mathbb{Z}_{N_{\rm f}})_L$ and $SU(N_{\rm f})/\mathbb{Z}_{N_{\rm f}}$ gauge field:

$$\partial_\mu J^\mu_B = \frac{N_{\rm f}}{(2\pi)^2} {\rm d}A_\chi \wedge B_{\rm f}.$$

Skyrmions in ordinary chiral SSB

Let us see how the discrete anomaly is matched for ordinary case:

$$G = \frac{SU(N_{\rm f})_L \times SU(N_{\rm f})_R \times U(1)_V}{\mathbb{Z}_{N_{\rm c}} \times \mathbb{Z}_{N_{\rm f}}} \to H = \frac{SU(N_{\rm f})_V \times U(1)_V}{\mathbb{Z}_{N_{\rm c}} \times \mathbb{Z}_{N_{\rm f}}}$$

The nonlinear Langrangian has the target space $G/H = SU(N_{\rm f})$.

Since $\pi_3(G/H) = \mathbb{Z}$, there are Skyrmions with the skyrmion current

$$J_{\rm skyrmion} = \frac{1}{24\pi^2}{\rm tr}[(U^\dagger {\rm d}U)^3].$$

$\Rightarrow {\rm d}J_{\rm skyrmion}[A_\chi, B_{\rm f}] = {\rm d}J_B[A_\chi, B_{\rm f}]$, and the anomaly matching is satisfied. (1807.07666[hep-th])

This is an extension of the $U(1)_V$-$SU(N_{\rm f})_L$-$SU(N_{\rm f})_L$ anomaly matching in the chiral Lagrangian. (ref. Witten, 1983)

Exotic chiral symmetry breaking

We now consider the Stern phase, where

$$G \to G^{\rm sub} = \frac{SU(N_F)_V \times U(1)_V}{(\mathbb{Z}_{N_{\rm c}}) \times (\mathbb{Z}_{N_{\rm f}})} \times (\mathbb{Z}_{N_{\rm f}})_L.$$

The target space of the nonlinear Lagrangian is

$$G/G^{\rm sub} = SU(N_{\rm f})/\mathbb{Z}_{N_{\rm f}}.$$

$\Rightarrow$ We can obtain the effective theory by gauging $\mathbb{Z}_{N_{\rm f}}$ in the ordinary chiral Lagrangian.

U : Nonlinear sigma field $\in SU(N_{\rm f})$, $a_\chi : \mathbb{Z}_{N_{\rm f}}$ dynamical gauge field

Mismatch of anomaly in Stern phase

Nontrivial homotopy: $\pi_3(G/G^{\rm sub}) = \mathbb{Z}$, $\pi_1(G/G^{\rm sub}) = \mathbb{Z}_{N_{\rm f}}$.
There are skyrmions with the current $J_{\rm skyrmion}$. There are also $\mathbb{Z}_{N_{\rm f}}$ domain walls.

Under the background gauge fields, the anomalous violation of $J_{\rm skyrmion}$ is

$$\mathrm{d}J_{\rm skyrmion} = \frac{N_{\rm f}}{(2\pi)^2}\mathrm{d}a_\chi \wedge B_{\rm f} \neq \mathrm{d}J_B.$$

Therefore, the anomaly matching is not satisfied.

$\Rightarrow$ We rule out the Stern phase from the possible QCD vacua.
This proof is valid even at finite densities. (1807.07666[hep-th])

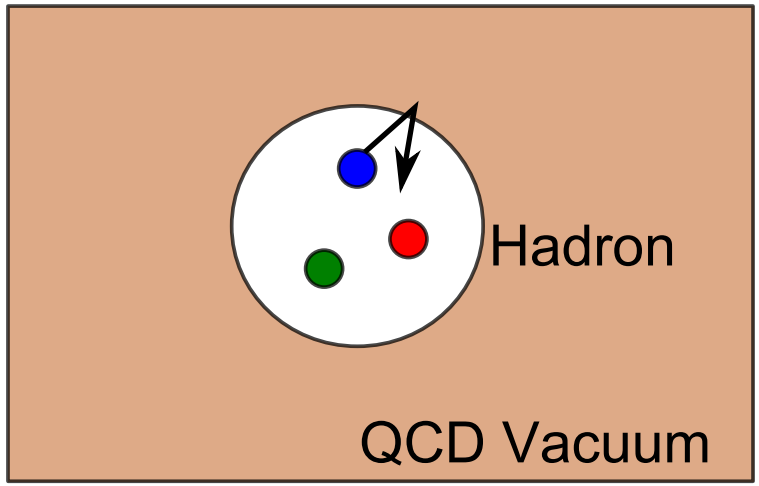
Application 2. Confinement and chiral symmetry, $\mathbb{Z}_3$-QCD

(YT, Misumi, Sakai, 1710.08923, YT, Kikuchi, Misumi, Sakai, 1711.10487)

Confinement $\Rightarrow$ Chiral symmetry breaking?

Heuristic argument (like the bag model) indicates that

$$\text{Confinement} \Rightarrow \text{Chiral symmetry breaking}$$



Can we make the EXACT statement related to this problem based on QCD?

Center symmetry and fundamental quarks

Center symmetry is explicitly broken with fundamental quarks
$\Rightarrow$ Let's consider a nice setup to have a well-defined question ($\mathbb{Z}_{N_C}$-QCD (Kouno et al. 2012, ...)).

Prepare $N_F = N_C$ quarks, and put the boundary condition on the quark field as $(f = 1, \ldots, N)$

$$q_f(\boldsymbol{x}, x_4 + L) = \omega^f q_f(\boldsymbol{x}, x_4).$$

This theory has the $(\mathbb{Z}_N)_{\text{shift,center}}$ symmetry, defined by

$$\Phi = \mathrm{tr_c}\left[P \exp\left(\mathrm{i}\int_{S^1} a\right)\right] \mapsto \omega\,\Phi,\ q_f \mapsto q_{f+1}.$$

Theorem

For $\mathbb{Z}_N$-QCD, confinement $\Rightarrow$ Chiral symmetry breaking.

Discrete anomaly of massless QCD

We show that massless N-flavor QCD has a mixed 't Hooft anomaly between

$$\frac{SU(N_F)_V \times U(1)_V}{(\mathbb{Z}_{N_C}) \times (\mathbb{Z}_{N_F})} \quad \text{and} \quad (\mathbb{Z}_{2N})_{\text{axial}}.$$

The anomaly is given as ($N_C = N_F = N$), under discrete axial trans.,

$$\mathcal{Z}[(A_{\rm f}, A_{\rm V}, B_{\rm c}, B_{\rm f})] \mapsto \mathcal{Z}[(A_{\rm f}, A_{\rm V}, B_{\rm c}, B_{\rm f})] \exp\left(-\frac{{\rm i}N}{2\pi}\int B_{\rm c} \wedge B_{\rm f}\right).$$

We will discuss consequences of this anomaly.

Anomaly and S^1 compactification

Remark: 't Hooft anomaly does not constrain S^1-compactified QFTs.

Example: Consider 3D Dirac fermion, $\overline{\psi}\sigma_i\partial_i\psi$. Symmetry $= U(1) \times \mathsf{T}$

$$\mathsf{T} : Z[A] \mapsto Z[A] \exp\left(\frac{\mathrm{i}}{4\pi}\int A\mathrm{d}A\right).$$

At finite temperatures, all the fermions get KK mass, $\frac{2\pi}{\beta}(n+1/2)$; the system is gapped and has the unique ground state.

Theorem

Despite this fact, for massless $\mathbb{Z}_N$-QCD, the new anomaly $\mathcal{A} = -\frac{N}{2\pi}\int B_{\mathrm{c}} \wedge B_{\mathrm{f}}$ of massless QCD still gives constraints.

(Tanizaki, Misumi, Sakai, 1710.08923)

Symmetry and Anomaly of massless $\mathbb{Z}_N$-QCD

There are three symmetries for massless $\mathbb{Z}_N$-QCD as a 3D QFT:

$$(\mathbb{Z}_N)_{\text{shift,center}}, \quad \frac{U(1)_F^{N-1} \times U(1)_V}{(\mathbb{Z}_N)_C \times (\mathbb{Z}_N)_F}, \quad \text{and} \quad (\mathbb{Z}_{2N})_{\text{axial}}$$

We can gauge $(\mathbb{Z}_N)_{\text{shift}}$ and $[U(1)_F^{N-1} \times U(1)_V]/[(\mathbb{Z}_N)_C \times (\mathbb{Z}_N)_F]$ by introducing the following two-form gauge fields (+ ordinary ones):

$$B_{\text{c}} = B_{\text{c}}^{(1)} \wedge L^{-1} \mathrm{d}x^4 + B_{\text{c}}^{(2)}, \quad B_{\text{f}} = B_{\text{f}}^{(2)}.$$

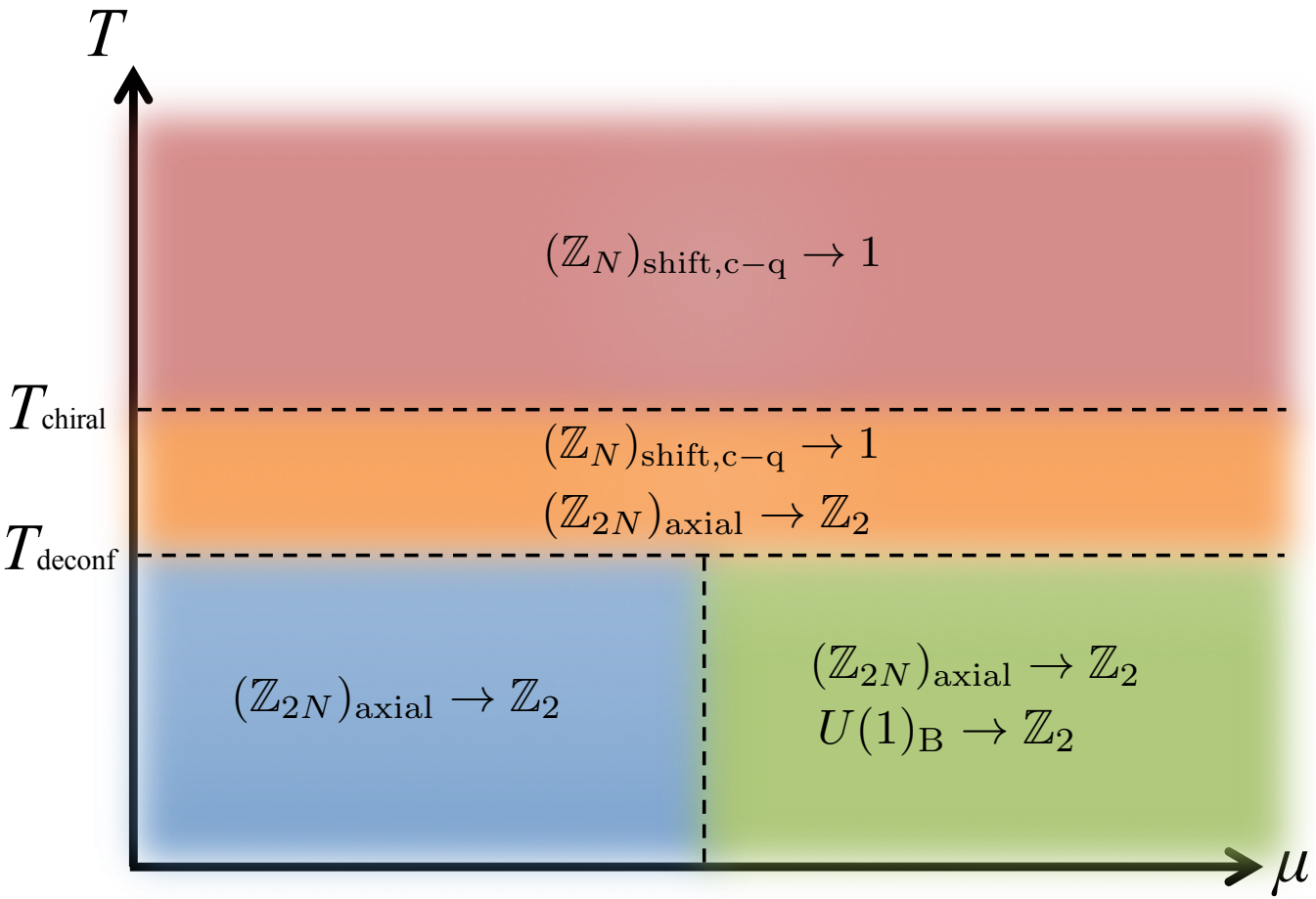
Substituting it into the four-dimensional anomaly, we obtain the anomaly for massless $\mathbb{Z}_N$-QCD:

$$\mathcal{A} = -\frac{N}{2\pi} \int B_{\text{c}}^{(1)} \wedge B_{\text{f}}^{(2)} \in \frac{2\pi}{N} \mathbb{Z}.$$

(1710.08923, 1711.10487)

One of possible phase diagrams

Anomaly matching: the phase is nontrivial at any T and μ (1711.10487):



Result ($\mu = 0$) looks consistent with lattice (Iritani, Itou, Misumi, 1508.07132).

Application 3. $SU(N)$ spin chain and generalized Haldane conjecture

(YT, Sulejmanpasic, 1805.11423)

$SU(2)$ spin chains and Haldane conjecture

Consider the 1D quantum antiferromagnetic spin chain ($J > 0$):

$$H = J \sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}.$$

Haldane ('83) showed that the low-energy EFT is the $\mathbb{CP}^1$ NLσM,

$$S = \frac{1}{g^2} \int |(\mathrm{d} + \mathrm{i}a)z|^2 + \frac{\mathrm{i}\theta}{2\pi} \int \mathrm{d}a,$$

with $\theta = 2\pi|\boldsymbol{S}|$ (as $|\boldsymbol{S}| \gg 1$).

- $S = 1, 2, 3, \ldots \Rightarrow$ The system has the mass gap (Polyakov, '75).
- $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \Rightarrow$ The system is gappless, especially described by $SU(2)_1$ WZW model (Affleck, Lieb, '86, Affleck, Haldane, '87)

  (cf. Komargodski, Sharon, Thorngren, Zhou, '17).

Generalization of the Haldane conjecture to $SU(N)$

Consider the $SU(N)$ spin chain with p-box rep. on each site.

Nonlinear sigma model

Under certain assumptions, low-energy EFT is $SU(N)/U(1)^{N-1}$

NLσM (Bykov ('12,'13), Lajko, Wamer, Mila, Affleck ('17)):

$$S = \frac{1}{g^2}\sum_{i=1}^{N}\int |(\mathrm{d} + \mathrm{i}a_i)z_i|^2 + \mathrm{i}\sum_{i=1}^{N}\frac{\theta_i}{2\pi}\int \mathrm{d}a_i + \cdots .$$

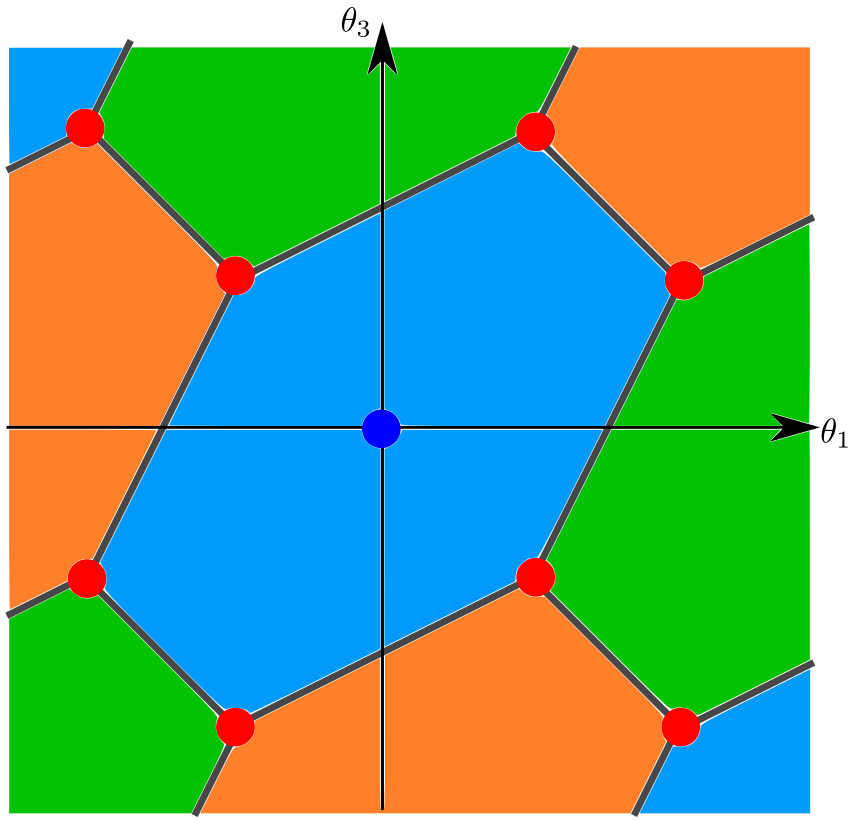
Here, $U = [z_1, \ldots, z_N]$ is an $SU(N)$ matrix, and

$$\theta_k = \frac{2\pi p}{N}k.$$

Using this result, we constrain the phase diagram by anomaly and global inconsistency matching, and generalize the Haldane conjecture.

Phase diagram of $SU(3)/U(1)^2$ NLσM

Before showing the computation, let us show the result:



- Different colors = different SPT phases (by global inconsistency)
- Red blobs = $SU(3)_1$ WZW model, or trimerized phase (by anomaly matching)

Symmetries of $SU(3)/U(1)^2$ NLσM

$PSU(3)$ spin symmetry

$z_i \mapsto V z_i$ with $V \in SU(3)$, coming from the spin rotation $\boldsymbol{S}_i \mapsto V \boldsymbol{S}_i V^{-1}$.

$\mathbb{Z}_3$ permutation symmetry

$z_i \mapsto z_{i+1}$ and $a_i \mapsto a_{i+1}$, coming from lattice translation $\boldsymbol{S}_i \mapsto \boldsymbol{S}_{i+1}$.
This symmetry exists only for special theta angles, such as $\theta_i = \frac{2\pi p}{3} i$.

Parity, P_k

$\theta_i \mapsto -\theta_{k-i}$, coming from $\boldsymbol{S}_i \mapsto \boldsymbol{S}_{k-i}$.
This exists only for special theta's, satisfying $\theta_i + \theta_{k-i} = 0 \mod 2\pi$.
(Gray lines in the Figure)

Gauging $PSU(3)$ symmetry

We will see that the spin rotation and lattice translation cannot be gauged simultaneously ($=$ mixed 't Hooft anomaly).

To gauge $PSU(3)$, we introduce (cf. Kapustin, Seiberg, 2014)

- A: $U(3)$ gauge field,
- B: $U(1)$ two-form gauge field, with $3B = \mathrm{d}(\mathrm{tr}(A))$.

The gauged action is given by

$$S[(A,B)] = \frac{1}{g^2}\sum_{i=1}^{3}\int |(\mathrm{d}+\mathrm{i}a_i+\mathrm{i}A)z_i|^2 + \mathrm{i}\sum_{i=1}^{3}\frac{\theta_i}{2\pi}\int(\mathrm{d}a_i+B)+\cdots .$$

$\Rightarrow$ 2π periodicity of theta's becomes $2\pi N(=6\pi)$.

$PSU(3)$-$\mathbb{Z}_3$ 't Hooft anomaly

Define the partition function $Z[A, B]$ with the $PSU(3)$ gauge field.

$\mathbb{Z}_3$ lattice translation gives

$$Z[A, B] \mapsto Z[A, B] \exp\left(\mathrm{i}p \int B\right).$$

This phase is nontrivial iff $p \neq 0$ mod 3, and gives 't Hooft anomaly.
't Hooft anomaly matching is satisfied by

- SSB of $\mathbb{Z}_3$ lattice translation (trimerized phase), or
- Conformal field theory.

Second choice gives the $SU(3)$ version of the Haldane conjecture.

(Affleck, Lieb, '86, Lajko, Wamer, Mila, Affleck '17, YT, Sulejmanpasic, 1805.11423)

$SU(3)$ Wess-Zumino-Witten model

The level-k $SU(3)$ WZW model is defined by

$$S = \frac{k}{8\pi^2}\int_{M_2} |\mathrm{d}\mathcal{U}|^2 + \frac{\mathrm{i}k}{12\pi}\int_{M_3} \mathrm{tr}[(\mathcal{U}^{-1}\mathrm{d}\mathcal{U})^3].$$

The model has the chiral symmetry, and we pay attention to its subgroup:

$$\frac{SU(3)_L \times SU(3)_R}{\mathbb{Z}_3} \supset PSU(3)_V \times (\mathbb{Z}_3)_L.$$

We can show that it has the same anomaly of $SU(3)/U(1)^2$ model if

$$\gcd(3,p) = \gcd(3,k).$$

$\Rightarrow$ Combined with the C-theorem, we conjecture that the anomaly is matched by $SU(3)_1$ WZW. (YT, Sulejmanpasic, 1805.11423)

(cf. Yao, Hsieh, Oshikawa, '18, Ohmori, Seiberg, Shao, '18)

$PSU(3)$-P global inconsistency

We take two parity symmetry point, such as $(\theta_1, \theta_3) = (0, 0)$ and $(2\pi, 0)$. Then,

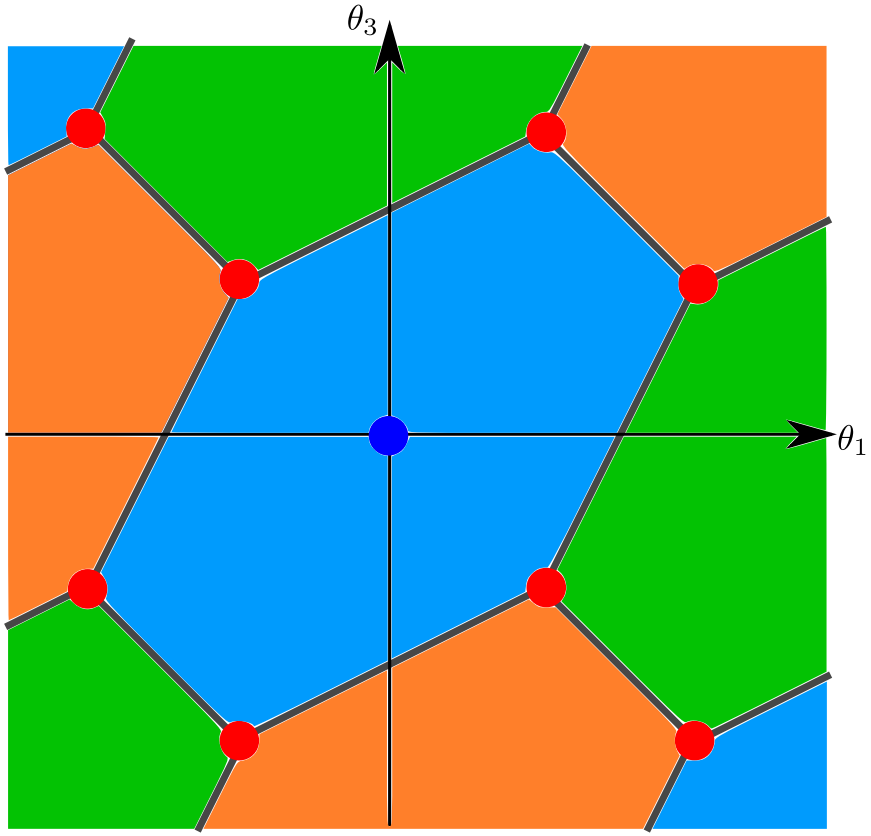
$$\begin{aligned} \mathsf{P} \quad : \quad & Z_{(0,0)}[A, B] \mapsto Z_{(0,0)}[A, B], \\ & Z_{(2\pi,0)}[A, B] \mapsto Z_{(2\pi,0)}[A, B] \exp\left(-2\mathrm{i} \int B\right). \end{aligned}$$

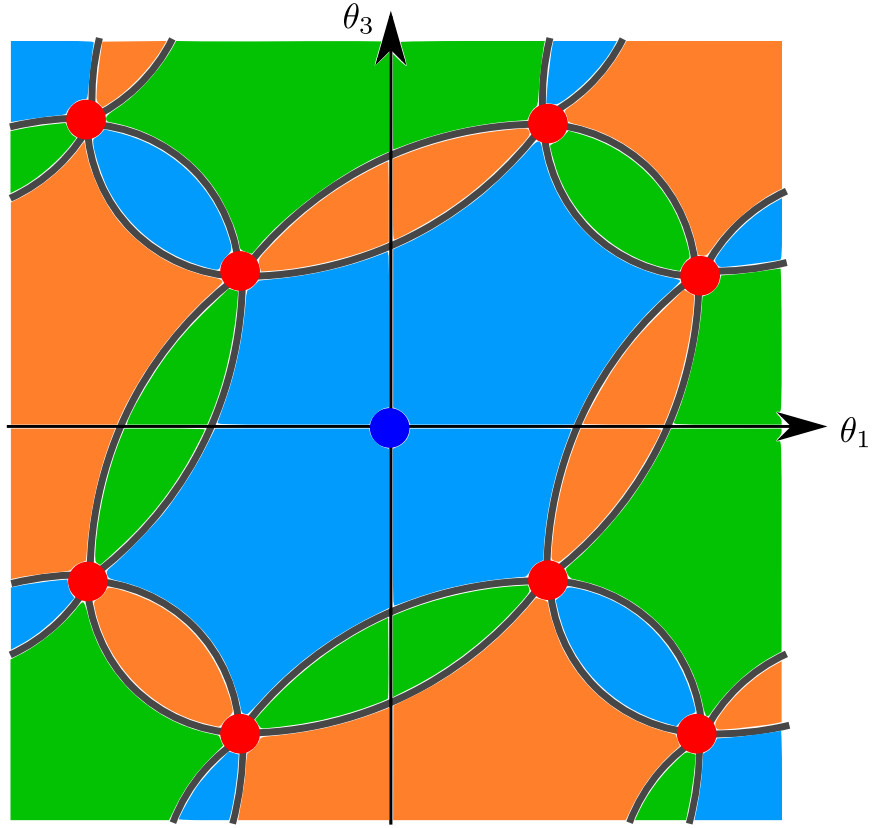
The second one is not anomaly, because we can cancel it by a counter term $\mathrm{i} \int B$.

This is the global inconsistency, i.e. we cannot gauge P for both points with the same counterterm (Gaiotto, Kapustin, Komargodski, Seiberg, '17). This is matched by (Tanizaki, Kikuchi, '17)

- Those two points are different SPT orders protected by $PSU(3)$.
- One of them has nontrivial ground states.

Two possible phase diagrams of $SU(3)/U(1)^2$ NLσM





- Different colors = different SPT phases (by global inconsistency)
- Red blobs = $SU(3)_1$ WZW model, or trimerized phase (by anomaly matching)

(YT, Sulejmanpasic, 1805.11423)

Summary

- It is EASY to show non-existence of unique gapped ground state.
- Important tool for it is anomaly matching.
- Nonperturbative aspects of QFT can be uncovered rigorously!